MOTION OF A SPHERE IN FLUID CAUSED BY VIBRATIONS OF ANOTHER SPHERE

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The following problem is considered in this work. There are two hard spheres immersed in an ideal, incompressible fluid, unbounded from outside. Initially, the fluid and the spheres are at rest with regard to an inertial system of orthogonal coordinates x, y, z. Subsequently, one of the spheres executes given periodical vibrations and its center moves along the y-axis. The center of the other sphere, identical with its center of mass, is located in the plane z = 0 that contains the y-axis and the point where its center was located at the initial time. Only the fluid pressure forces act on the second sphere (the z-component of the total force acting on the second sphere is equal to zero). The fluid flow is potential. The problem is to determine the motion of the second sphere. Under the assumption that the initial distance between the centers of the spheres is large compared to their radii and to the largest displacement of the center of the first sphere from its initial position, it is shown that the second sphere moves away from the first sphere when the average density of the second sphere is smaller than the fluid density, and moves toward the first sphere when its average density is larger than the fluid density.

Let t denote time; T is the period of vibrations of the center of the first sphere; H = (0, H, 0) is the position vector of the center of the first sphere;

$$H = A_0 + \sum_{m=1}^{\infty}\left(A_m \cos 2m\pi\frac{t}{T} + B_m \sin 2m\pi\frac{t}{T}\right)$$

(A_0, A_m, B_m are constant); R = (X, Y, 0) is the position vector of the center of the second sphere; X_0, Y_0 are the values of X, Y for t = 0; Φ is the potential of the flow velocity of the fluid; m_1, m_2 are the masses of the spheres; S_1, S_2 are their surfaces; η is the external unit normal to the surface; and ρ_{fl} is the fluid density.

The coordinates X, Y, and the potential Φ satisfy the following equations and conditions

$$\frac{d}{dt}\frac{\partial E}{\partial \dot{X}} - \frac{\partial E}{\partial X} = 0; \tag{1}$$

$$\frac{d}{dt}\frac{\partial E}{\partial \dot{Y}} - \frac{\partial E}{\partial Y} = 0; \tag{2}$$

$$X = X_0,\ Y = Y_0,\ \dot{X} = 0,\ \dot{Y} = 0 \ \text{at} \quad t = 0; \tag{3}$$

$$\Delta\Phi = 0; \tag{4}$$

$$\mathbf{n}\cdot\nabla\Phi = \mathbf{n}\cdot\dot{\mathbf{H}} \ \text{on} \ S_1, \tag{5}$$

$$\mathbf{n}\cdot\nabla\Phi = \mathbf{n}\cdot\dot{\mathbf{R}} \ \text{on} \ S_2; \tag{6}$$

$$\nabla\Phi \to 0 \ \text{for} \quad x^2 + y^2 + z^2 \to \infty, \tag{7}$$

where

$$E = \frac{1}{2} m_1 \dot{H}^2 + \frac{1}{2} m_2 (\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\rho_{fl} \iiint\limits_{\Omega} (\nabla\Phi)^2\, dx\, dy\, dz \tag{8}$$

is the sum of kinetic energies of the spheres and the fluid (Ω is the region occupied by the fluid). The coordinate H satisfies the conditions

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$$H=0,\ \dot{H}=0 \text{ at } t=0. \tag{9}$$

Eqs. (1), (2) are the Lagrange equations [1-3], also called the Thomson–Tait equations [3].

It is now assumed that the initial distance L_0 between the centers of the spheres is large compared to their radii a_1, a_2 and the largest value of $|H|$. The number of $\alpha = a_1/L$ is employed as a small parameter.

The problem of potential flow of an ideal, incompressible fluid, caused by a given motion of two hard spheres immersed in it is discussed in [4]. Employing the method of calculation of the flow velocity potential introduced in [4], we find the solution of the problem (4), (7) satisfying the conditions (5), (6) that is accurate in the quantities proportional to $\dot{H}$, $\dot{X}$, $\dot{Y}$ and small compared to $\alpha^7\dot{H}$, $\alpha^7\dot{X}$, $\alpha^7\dot{Y}$, respectively. Using this solution of the problem (4)-(7) and Eq. (8) we obtain an approximate expression for E

$$E=\frac{1}{2}\rho_{fl}V\left(A_{HH}\dot{H}^2+A_{HX}\dot{H}\dot{X}+A_{HY}\dot{H}\dot{Y}+A_{XX}\dot{X}^2+\right.$$
$$\left.+A_{XY}\dot{X}\dot{Y}+A_{YY}\dot{Y}^2\right), \tag{10}$$

$$V=\frac{4}{3}\pi a_2^3;\quad A_{HH}=\frac{m_1}{\rho_{fl}V}+\frac{a_1^3}{2a_2^3}+\frac{3}{8}\sum_{m=0}^{1}\frac{(-1)^m}{m!}\alpha^{m+6}h^mL_0^{m+6}\frac{\partial^m}{\partial Y^m}\frac{X^2+4Y^2}{R^8};$$

$$A_{HX}=\frac{9}{2}\frac{X}{L_0}\sum_{m=0}^{4}\frac{(-1)^{m+1}}{m!}\alpha^{m+3}h^mL_0^{m+4}\frac{\partial^m}{\partial Y^m}\frac{Y}{R^5};$$

$$A_{HY}=\frac{3}{2}\sum_{m=0}^{4}\frac{(-1)^m}{m!}\alpha^{m+3}h^mL_0^{m+3}\frac{\partial^m}{\partial Y^m}\frac{X^2-2Y^2}{R^5};$$

$$A_{XX}=\frac{\rho_{sph}}{\rho_{fl}}+\frac{1}{2}+\frac{3}{8}\frac{a_2^3}{a_1^3}\sum_{m=0}^{1}\frac{(-1)^m}{m!}\alpha^{m+6}h^mL_0^{m+6}\frac{\partial^m}{\partial Y^m}\frac{4X^2+Y^2}{R^8};$$

$$A_{XY}=\frac{9}{4}\frac{a_2^3}{a_1^3}\frac{X}{L_0}\sum_{m=0}^{1}\frac{(-1)^m}{m!}\alpha^{m+6}h^mL_0^{m+7}\frac{\partial^m}{\partial Y^m}\frac{Y}{R^8};$$

$$A_{YY}=\frac{\rho_{sph}}{\rho_{fl}}+\frac{1}{2}+\frac{3}{8}\frac{a_2^3}{a_1^3}\sum_{m=0}^{1}\frac{(-1)^m}{m!}\alpha^{m+6}h^mL_0^{m+6}\frac{\partial^m}{\partial Y^m}\frac{X^2+4Y^2}{R^8}$$

where $h = H/a_1$, $R=\sqrt{X^2+Y^2}$, $\rho_{sph} = m_2/V$ is the average density of the second sphere. Substituting expression (10) for E into (1), (2) we find the following approximate equations for X, Y:

$$\frac{d^2}{d\tau^2}\frac{X}{L_0}+\lambda\frac{X}{L_0}\sum_{m=0}^{4}\frac{(-1)^{m+1}}{(m+1)!}\alpha^{m+4}\frac{d^2h^{m+1}}{d\tau^2}L_0^{m+4}\frac{\partial^m}{\partial Y^m}\frac{Y}{R^5}+$$
$$+\frac{1}{2}\alpha^8\lambda\left(\frac{dh}{d\tau}\right)^2L_0^7\frac{X(X^2+5Y^2)}{R^{10}}+Q_X=0, \tag{11}$$

$$\frac{d^2}{d\tau^2}\frac{Y}{L_0}+\frac{1}{3}\lambda\sum_{m=0}^{4}\frac{(-1)^m}{(m+1)!}\alpha^{m+4}\frac{d^2h^{m+1}}{d\tau^2}L_0^{m+3}\frac{\partial^m}{\partial Y^m}\frac{X^2-2Y^2}{R^5}+$$
$$+2\alpha^8\lambda\left(\frac{dh}{d\tau}\right)^2L_0^7\frac{Y^3}{R^{10}}+Q_Y=0,$$

where $\tau = t/T$; $\lambda = 9\rho_{fl}/[4(\rho_{sph}+\rho_{fl}/2)]$;

$$Q_X=\frac{2\lambda}{9L_0}\left[\left(2A_{XX}-\frac{9}{2\lambda}\right)\frac{d^2X}{d\tau^2}+A_{XY}\frac{d^2Y}{d\tau^2}+\frac{3}{8}\alpha^9h\left(\frac{dh}{d\tau}\right)^2L_0^9\frac{\partial^2}{\partial X\partial Y}\frac{X^2+4Y^2}{R^8}+\right.$$
$$+2\frac{\partial A_{XX}}{\partial h}\frac{dh}{d\tau}\frac{dX}{d\tau}+\frac{\partial A_{XY}}{\partial h}\frac{dh}{d\tau}\frac{dY}{d\tau}+\frac{\partial A_{XX}}{\partial X}\left(\frac{dX}{d\tau}\right)^2+2\frac{\partial A_{XX}}{\partial Y}\frac{dX}{d\tau}\frac{dY}{d\tau}+$$
$$\left.+\left(\frac{\partial A_{XY}}{\partial Y}-\frac{\partial A_{YY}}{\partial X}\right)\left(\frac{dY}{d\tau}\right)^2\right];$$

$$Q_Y=\frac{2\lambda}{9L_0}\left[A_{XY}\frac{d^2X}{d\tau^2}+\left(2A_{YY}-\frac{9}{2\lambda}\right)\frac{d^2Y}{d\tau^2}+\frac{3}{8}\alpha^9h\left(\frac{dh}{d\tau}\right)^2L_0^9\frac{\partial^2}{\partial Y^2}\frac{X^2+4Y^2}{R^8}+\right.$$

$$+\frac{\partial A_{XY}}{\partial h}\frac{dh}{d\tau}\frac{dX}{d\tau}+2\frac{\partial A_{YY}}{\partial h}\frac{dh}{d\tau}\frac{dY}{d\tau}+\left(\frac{\partial A_{XY}}{\partial X}-\frac{\partial A_{XX}}{\partial Y}\right)\left(\frac{dX}{d\tau}\right)^2+$$
$$+2\frac{\partial A_{YY}}{\partial X}\frac{dX}{d\tau}\frac{dY}{d\tau}+\frac{\partial A_{YY}}{\partial Y}\left(\frac{dY}{d\tau}\right)^2\Bigg].$$

Now we apply the method of averaging [5, 6]. Let η, ξ be variables related to X, Y by the equations

$$\frac{X}{L_0}=\eta+\lambda\eta\sum_{m=0}^{4}\frac{(-1)^m}{(m+1)!}\alpha^{m+4}h^{m+1}\frac{\partial^m}{\partial\xi^m}\frac{\xi}{(\eta^2+\xi^2)^{5/2}},\qquad(12)$$
$$\frac{Y}{L_0}=\xi+\frac{1}{3}\lambda\sum_{m=0}^{4}\frac{(-1)^{m+1}}{(m+1)!}\alpha^{m+4}h^{m+1}\frac{\partial^m}{\partial\xi^m}\frac{\eta^2-2\xi^2}{(\eta^2+\xi^2)^{5/2}};$$

$$\chi=\alpha^{-4}d\eta/d\tau,\ \psi=\alpha^{-4}d\xi/d\tau.\qquad(13)$$

According to (8), (9), (12), (13) η, ξ, χ, ψ satisfy the conditions

$$\eta=\eta_0,\ \xi=\xi_0,\ \chi=0,\ \psi=0\ \text{ at }\ \tau=0,\qquad(14)$$

where $\eta_0 = X_0/L_0$; $\xi_0 = Y_0/L_0$. Using (12), (13) we bring (11) to the system of equations for η, ξ, χ, ψ in normal form. Expanding the right-hand sides of these equations in powers of α and keeping only the largest terms of the expansion, we obtain the following system of equations in the standard form

$$d\eta/d\tau=\alpha^4\chi,\ d\xi/d\tau=\alpha^4\psi;\qquad(15)$$
$$\frac{d\chi}{d\tau}=-\frac{1}{2}\alpha^4\lambda\left\{\left[\left(\frac{dh}{d\tau}\right)^2+\frac{2}{3}\lambda h\frac{d^2h}{d\tau^2}\right]\frac{\eta(\eta^2+5\xi^2)}{\sigma^{10}}+4\frac{dh}{d\tau}\left(\chi\frac{\partial}{\partial\eta}+\psi\frac{\partial}{\partial\xi}\right)\frac{\eta\xi}{\sigma^5}\right\},$$
$$\frac{d\psi}{d\tau}=-2\alpha^4\lambda\left\{\left[\left(\frac{dh}{d\tau}\right)^2+\frac{2}{3}\lambda h\frac{d^2h}{d\tau^2}\right]\frac{\xi^3}{\sigma^{10}}-\frac{1}{3}\frac{dh}{d\tau}\left(\chi\frac{\partial}{\partial\eta}+\psi\frac{\partial}{\partial\xi}\right)\frac{\eta^2-2\xi^2}{\sigma^5}\right\},$$

where $\sigma=\sqrt{\eta^2+\xi^2}$. We carry out the average of (15) over the dimensionless time τ explicitly entering the formula. As a result we obtain

$$d\eta/d\tau=\alpha^4\chi,\ d\xi/d\tau=\alpha^4\psi,\qquad(16)$$
$$d\chi/d\tau=-\alpha^4\varkappa\lambda k\eta(\eta^2+5\xi^2)/\sigma^{10},$$
$$d\psi/d\tau=-4\alpha^4\varkappa\lambda k\xi^3/\sigma^{10},$$

where $\varkappa=(\rho_{sph}-\rho_{fl})/(\rho_{sph}+\rho_{fl}/2)$; $k=\frac{\pi^2}{a_1^2}\sum_{m=1}^{\infty}m^2(A_m^2+B_m^2)$.

In accordance with (14), (16) we have

$$d^2\eta/d\tau^2=-\alpha^8\varkappa\lambda k\eta(\eta^2+5\xi^2)/\sigma^{10},\qquad(17)$$
$$d^2\xi/d\tau^2=-4\alpha^8\varkappa\lambda k\xi^3/\sigma^{10};$$

$$\eta=\eta_0,\ \xi=\xi_0,\ d\eta/d\tau=0,\ d\xi/d\tau=0\ \text{ at }\ \tau=0.\qquad(18)$$

It follows from (17), (18) that

$$\eta=\eta_0,\ \xi=\xi_0\ \text{ for }\ \varkappa=0.\qquad(19)$$

Let $\kappa \neq 0$. When $\eta_0 = 0$, $\xi_0 = \pm 1$ and $\eta_0 = \pm 1$, $\xi_0 = 0$ the solution of the problem (17), (18) is determined by the equations

$$\eta=0,\qquad(20)$$

$$\frac{\sqrt{3}}{2}\int_{\xi_0}^{\xi}\frac{du}{\sqrt{\frac{\varkappa}{|\varkappa|}(u^{-6}-1)}}=\begin{cases} s \text{ for } \varkappa<0, \ \xi_0=1\\ \text{and } \varkappa>0, \ \xi_0=-1,\\ -s \text{ for } \varkappa<0, \ \xi_0=-1\\ \text{and } \varkappa>0, \ \xi_0=1;\end{cases}$$

$$\sqrt{3}\int_{\eta_0}^{\eta}\frac{du}{\sqrt{\frac{\varkappa}{|\varkappa|}(u^{-6}-1)}}=\begin{cases} s \text{ for } \varkappa<0, \ \eta_0=1\\ \text{and } \varkappa>0, \ \eta_0=-1,\\ -s \text{ for } \varkappa<0, \ \eta_0=-1\\ \text{and } \varkappa>0, \ \eta_0=1,\end{cases} \qquad (21)$$

$$\xi=0,$$

where $s = \alpha\sqrt[4]{|\kappa|}\lambda k\tau$.

For values of s small compared with unity we find the following approximate solutions of the problem (17), (18):

$$\eta=\eta_0\left[1-\frac{1}{2}\frac{\varkappa}{|\varkappa|}(1+4\xi_0^2)s^2\right], \qquad (22)$$

$$\xi=\xi_0\left(1-2\frac{\varkappa}{|\varkappa|}\xi_0^2 s^3\right).$$

Using (22) we obtain an approximate expression for σ

$$\sigma=1-\frac{1}{2}\frac{\varkappa}{|\varkappa|}(1+3\xi_0^2)s^2. \qquad (23)$$

Figures 1-3 show diagrams of the dependence of η, ξ, σ on s, obtained by means of numerical solution of the problems (17), (18). The curves 1-5 correspond to $\varkappa<0$, $\eta_0=\cos(n\pi/12)$, $\xi_0=\sin(n\pi/12)$ $(n=1, 2, \ldots, 5)$, and the curves 6-10 correspond to $\kappa>0$, $\eta_0=\cos(n\pi/12)$, $\xi_0=\sin(n\pi/12)$ $(n = 1, 2, \ldots, 5)$. According to (17), (18) the change of signs of η_0, ξ_0 leads to the change of signs of η, ξ. Therefore the problem (17), (18) was solved numerically

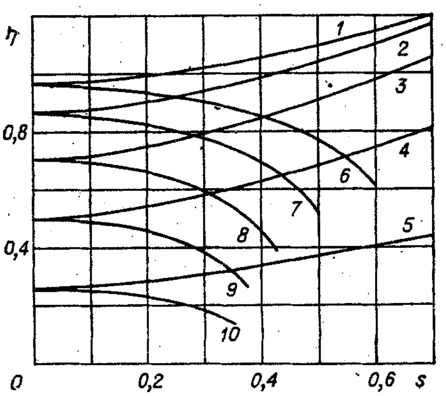

Fig. 1

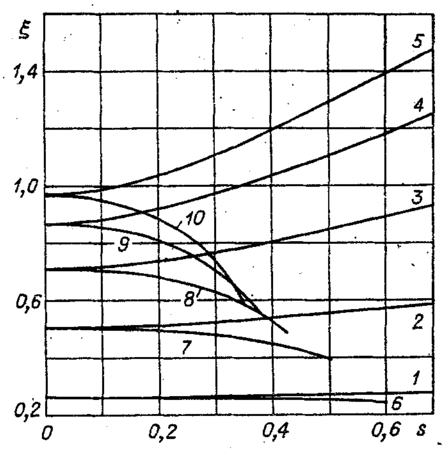

Fig. 2

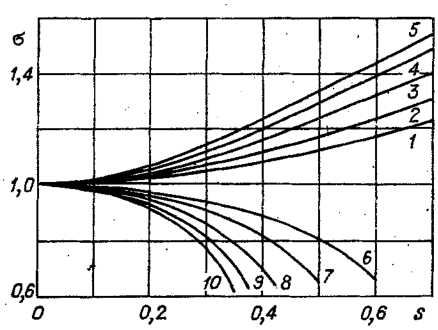

Fig. 3

only for positive signs of η_0, ξ_0. We note that the approximate solution (22) agrees with (20), (21) and with the results of numerical solution of (17), (18) when $s \lesssim 0.3$.

Using (12) we obtain the following approximate expressions for X, Y and the distance L between the centers of the spheres

$$X = L_0\eta,\ Y = L_0\xi; \tag{24}$$

$$L = L_0\sigma. \tag{25}$$

The data shown in Figs. 1, 2 and Eqs. (19)-(22), (24) approximately determine the dependence of X and Y on t. According to Eqs. (20), (21), (23), (25) and to the data shown in Fig. 3, when $\kappa < 0$ L increases, and for $\kappa > 0$ L decreases with increasing t. Thus the second sphere moves away from the first one when $\rho_{sph} < \rho_f$, and moves toward it when $\rho_{sph} > \rho_{fl}$.

LITERATURE CITED

1. M. Lamb, Hydrodynamics, Dover Publications, New York (1945).
2. L. M. Milne-Thompson, Theoretical Hydrodynamics, MacMillan, New York (1956).
3. V. L. Berdichevskii, Variational Principles in Mechanics of Continuous Media [in Russian], Nauka, Moscow (1983).
4. G. Kirchhoff, Mechanics [Russian translation], Akad. Nauk SSSR, Moscow (1962).
5. N. N. Bogolyubov, Yu. A. Mitropol'skii, Asymptotic Methods of the Theory of Nonlinear Vibrations [in Russian], Nauka, Moscow (1974).
6. Yu. A. Mitrpol'skii, Method of Averaging in Nonlinear Mechanics [in Russian], Naukova Dumka, Kiev (1971).